COMPARISON TECHNIQUE IN STABILITY THEORY
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1. INTRODUCTION

One of the main problems studied by Professor V. Lakshmikantham in his works was elaboration and further development of the principle of comparison with scalar or vector Liapunov function and its extension for the classes of systems other than ordinary differential equations. First he considered the differential system

\[
\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,
\]

(1.1)

where \( t \in \mathbb{R}_+ \) and \( f \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n) \) together with another system (or a differential equation)

\[
\frac{du}{dt} = g(t, u), \quad u(t_0) = u_0 \geq 0,
\]

(1.2)

where \( g \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m) \) \((g \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}_+^m, \mathbb{R}))\) and \( m \leq n \). If equation (1.2) is scalar then the inequality

\[
Dm(t) \leq g(t, m(t)),
\]

(1.3)

where \( Dm(t) \) is a fixed Dini derivative together with function \( v(t, x) \in \mathbb{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \), \( v(t, x) \) is locally Lipshitzian in \( x \), satisfying the inequality

\[
v(t + h, x + hf(t, x)) \leq v(t, x) + h(t, v(t, x)) + o(h),
\]

(1.4)
yields equation (1.2). This is a comparison equation for the initial system (1.1) and it allows one to investigate behavior of its solution without immediate integration. In the papers published during 1957-63 V. Lakshmikantham presented a systematic development of the theory of differential and integral inequalities for the \( \| x(t) \| \) or the function \( m(t) = v(t, x(t)) \) of Liapunov type. These results laid a basis of new approaches in qualitative theory of equations in both finite and infinite-dimensional Spaces (see [1]).

This paper presents survey of some results in motion stability theory for finite-dimensional systems obtained in context with the vector Liapunov function and comparison technique. Unlike many known results in the direction we use vector functions whose components are sign-definite with respect to a part of variables. This approach sometime enables us to simplify the construction of all appropriate vector Liapunov function.

2 VECTOR LIAPUNOV FUNCTIONS APPLICATIONS.

2.1 Scalar approach. We return back to the system (1.1) and consider also a vector function

\[ V(t,x) = (v_1(t,x), v_2(t,x), ..., v_m(t,x))^T, \]

where \( v_s \in C(R_+ \times R^n, R_+) \), \( s = 1, 2, ..., m \), and its total derivative along solutions of the system (1.1)

\[ D^s V(t,x) = \lim \sup \{ [V(t+\theta, x + \theta f(t,x)) - V(t,x)]\theta^{-1} : \theta \to 0^+ \} \]

for \( (t,x) \in R_+ \times R^n \).

The notion of the property of having a fixed sign of function (2.1) is introduced by means of one of the measures such as

(a) \( u_0(t,x) = e^T V(t, x) e, \ e = (1,1, ..., 1) \in R_m^1 \);

(b) \( v_0(t,x) = \alpha^T V(t, x) (t,x) \in R_+ \times R^n, \ \alpha \in R^m \);

(c) \( v_0(t,x) = \max_{1 \leq i \leq m} v_i(t,x) \);

(d) \( v_0(t,x) = Q(V(t,x)), \) where \( Q \in C(R^m_+, R_+), \ Q(0) = 0 \) and \( Q(u) \) is nondecreasing in \( u \).

The state vector \( x \) of system (1.1) will be divided into \( m \) subvectors, i.e. \( x = (x_1^T, ..., x_m^T)^T, \) where \( x_s \in R^n_s \) and \( n_1 + n_2 + ... + n_m = n \).
Assume that

\[ a_{i1} \psi_{i1}(||x||) \leq v_i(t,x) \leq a_{i2} \psi_{i2}(||x||), \quad i=1,2,\ldots,m, \quad (2.3) \]

where \( a_{i1} \) and \( a_{i2} \) are some positive constants and \( \psi_{i1} \) and \( \psi_{i2} \) are of class \( K \) (KR).

Actually the condition (2.3) means that the components \( v_i(t,x) \) of the vector function (2.1) are positive definite and decreasing with respect to a part of variables.

Let us introduce designations

\[ A_1 = \text{diag} \{ a_{11}, a_{12}, \ldots, a_{1m} \}, \]
\[ A_2 = \text{diag} \{ a_{21}, a_{22}, \ldots, a_{2m} \}, \quad (2.4) \]

\( v(t,x,\alpha) = \alpha^T V(t,x), \quad (t,x) \in R^+ \times R^n, \quad \alpha \in R^m. \)

**Proposition 2.1** For the vector function (2.1) to be positive definite and decreasing, it is sufficient that matrices (2.4) in the bilateral inequalities

\[ u_1^T A_1 u_1 \leq v(t,x,\alpha) \leq u_2^T A_2 u_2 \quad (2.5) \]

be positive definite, where

\[ u_1 = \left( \psi_{11}^{1/2}(||x_1||) \ldots \psi_{1m}^{1/2}(||x_m||) \right)^T, \]
\[ u_2 = \left( \psi_{21}^{1/2}(||x_1||) \ldots \psi_{2m}^{1/2}(||x_m||) \right)^T. \]

If \( \psi_{i1} = \psi_{i2} = ||x|| \), then the estimates (2.5) are known as the estimates characteristics of the quadratic forms.

Taking into account (2.2) we get for the function \( v(t,x,\alpha) \)

\[ D^* v(t,x,\alpha) = \alpha^T D^* V(t,x). \quad (2.6) \]

Let for \( (t,x) \in R^+ \times R^n \) there exist an \( m \times m \) matrix \( S(t,x) \), for which

\[ D^* v(t,x,\alpha) \leq \psi_3^T S(t,x) \psi_3, \quad (2.7) \]
where \( \psi_3 = \left( \psi_{13}^{1/2}(\|x_1\|), \psi_{23}^{1/2}(\|x_2\|), \ldots, \psi_{m3}^{1/2}(\|x_m\|) \right)^T \).

Estimates (2.5) -- (2.7) allow us to establish stability conditions for the state \( x = 0 \) of system (1.1) as follows.

**Theorem 2.1** Let the vector function \( f \) in system (1.1) be continuous on \( \mathbb{R}_+ \times \mathbb{N} \), \( f(t,0) = 0 \) for all \( t \in \mathbb{R}_+ \). If there exist

1. an open connected time-invariant neighbourhood \( G \) of point \( x = 0 \);
2. the decreasing positive definite vector function \( V \) on \( G \);
3. the \( m \times m \)-matrix \( S(t,x) \) on \( G \) such that inequality (2.7) is satisfied

then

(a) the state \( x = 0 \) of system (1.1) is uniformly stable if the matrix \( M(t,x) = (\frac{1}{2})(S^T(t,x) + S(t,x)) \) is negative semidefinite on \( G \);
(b) the state \( x = 0 \) of system (1.1) is uniformly asymptotically stable providing the matrix \( M(t,x) + \varepsilon E, \varepsilon > 0 \), \( E \) is \( m \times m \) identity matrix, is negative definite on \( G \);
(c) the state \( x = 0 \) of system (1.1) is exponentially stable if there exists constants \( C > 0 \) and \( b > 0 \) such that \( a\|x\|^b \leq u^T A u, \) function \( v(t, x, \alpha) \) is decrescent, and matrix \( M(t,x) + \varepsilon E \) is negative definite.

**Proof**: Formula (2.3) and estimates (2.5) and (2.7) allow us to repeat all points of the proof of classical theorems on uniform (asymptotic) stability.

**Remark 2.1** New points of Theorem 2.1 resulting from the application of vector function (2.1) are

(a) a possibility to apply the components \( v_i(t,x) \), \( i = 1, 2, \ldots, m \), being of a fixed sign with respect to a part of variables;
(b) a possibility to check the property of having a fixed sign of the function \( D^\dagger v(t,x,\alpha) \big|_{(1,1)} \) via the algebraic method.

The state \( (y = 0, z := 0) \) is asymptotically stable if and only if when the system of inequalities

\[
y^0_i > \frac{z}{\rho_i}, \quad y^0_i > 0, \quad i = 1, 2, \ldots, n,
\]

\[
2p - \sum_{i=1}^{n} \left| \frac{a_i}{\rho_i} \right| z^0 + g(z^0) > \sum_{i=1}^{n} \left| a_i \right| \rho_i y^0_i, \quad z^0 > 0,
\]

is joint.
After some transformations of this system we define an estimation

\[
\sum_{i=1}^{n} \frac{|a_i|}{p_i} \leq p
\]  

(4.3)

which determines restrictions on parameters for which the state \((y = 0, z = 0)\) is asymptotically stable.

**Remark 4.1:** For the systems of equations (4.2) for \(n = 4\) Pioptkovskii and Rutkovskaya (see [10]) determined the following estimation of the domain of parameters:

\[
\left( \min_i \frac{1}{p_i} \right)^\frac{3}{4} \sum_{i=1}^{3} |a_i| < \left( \frac{p}{4} \right)^3
\]  

(4.4)

It is obvious that for both \(n = 4\) and \(n > 4\) the estimation (4.3) defines a larger domain as compared with that ensured by the estimation (4.4).*

**REFERENCES**


* The author has proved some additional results which are not included here.

The interested readers will get a copy of the complete paper from the editors, if required.


