MONOTONE ITERATIVE TECHNIQUE FOR
PERIODIC BOUNDARY VALUE PROBLEM
ASSOCIATED WITH FRACTIONAL
DIFFERENCE EQUATIONS

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Abstract

Method of upper and lower solutions is an interesting and convenient technique for proving existence and uniqueness results for non-linear problems. This method coupled with monotone iterative technique is an efficient tool that offers theoretical existence results. In this paper we use upper and lower solutions along with monotone iterative technique to obtain the existence and uniqueness of solutions to periodic boundary value problems associated with fractional difference equations.

1 INTRODUCTION

Fractional calculus has gained importance during the last three decades due to its applicability in diverse fields of science and engineering such as visco-elasticity, neurology, control theory, statistics etc.[13]. The analogous theory of discrete fractional calculus was initiated and some basic properties of discrete fractional sums and difference operator have been established recently. In [1, 2, 3, 4], the authors established some basic fractional difference inequalities and comparison principles.

An interesting and useful technique for establishing existence and uniqueness results for non linear problems is the method of upper and lower solutions [9]. This method

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coupled with the monotone iterative technique manifests itself as an effective mechanism that offers existence results theoretically in a closed set.

In this paper, we use monotone iterative technique for Periodic Boundary Value Problem (PBVP) associated with fractional difference equation and derive the convergence of monotone sequences to maximal and minimal solutions using which the existence and uniqueness of solutions to PBVP associated with fractional difference equation are established.

We first present some basic definitions and results on discrete fractional calculus and difference equations

2 PRELIMINARIES

Definition 2.1 The backward difference operator $\triangle_{-n}$ is defined as $\triangle_{-n} = \varepsilon^{-1}(1-B)$ where $Bf(n) = f(n-1)$ is standard backward shift operator and $\varepsilon$ is interval length. Gray and Zhang [7] gave a definition of the fractional difference as follows:

Definition 2.2 For any complex number $\alpha$ and $f$ defined over the integer set $\{a-p, a-p+1, ..., n\}$, the $\alpha$th order difference of $f(n)$ over $N^+_a = \{a, a+1, ..., n\}$ is defined by

$$∇^\alpha f(n) = \frac{∇^p}{Γ(p-\alpha)} \sum_{k=0}^{n-\alpha} \frac{Γ(k+p-\alpha)}{Γ(k+1)} f(n-k) \tag{2.1}$$

Later Hirota [8] took the first $n$ terms of Taylor series of $\triangle_{-n} = \varepsilon^{-\alpha}(1-B)^\alpha$ and gave the following definition.

Definition 2.3 Let $\alpha \in \mathbb{R}$. Then difference operator of order $\alpha$ is defined by

$$\Delta_{-n}^\alpha u(n) = \left\{ \begin{array}{ll}
\varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u(n-j), & \alpha \neq 1, 2, ... \\
\varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u(n-j), & \alpha = m = 1, 2, 3, ...
\end{array} \right. \tag{2.2}$$

Here $\binom{n}{m}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \left\{ \begin{array}{ll}
\frac{Γ(a+1)}{Γ(a-n+1)Γ(n+1)} & n > 0 \\
1 & n = 0 \\
0 & n < 0.
\end{array} \right. \tag{2.3}$$

In 2002, Atsushi Nagai [11] introduced another definition of fractional difference which is a slight modification of Hirota’s fractional difference operator.

Definition 2.4 Let $\alpha \in \mathbb{R}$ and $m$ be an integer such that $m-1 < \alpha \leq m$. The difference operator $Δ_{-n}^\alpha$ of order $\alpha$ is defined as

$$Δ_{-n}^\alpha u(n) = Δ_{-n}^{α-m} Δ_{-n}^m u(n) = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{α-m}{j} (-1)^j Δ_{-n}(n-j) u(n-j). \tag{2.4}$$
By taking the interval length $\varepsilon = 1$, we have
\[
\nabla^\alpha u(n) = \sum_{j=0}^{n-1} \binom{\alpha - m}{j} (-1)^j \nabla^m u(n - j).
\] (2.5)

We know that for $0 < \alpha < 1$, 
\[
\binom{\alpha - 1}{j} = (-1)^j \binom{j - \alpha}{j}
\]
Now for $m = 1$, (2.5) becomes
\[
\nabla^\alpha u(n) = \sum_{j=0}^{n-1} \binom{\alpha - 1}{j} (-1)^j \nabla u(n - j) = \sum_{j=0}^{n-1} \binom{j - \alpha}{j} \nabla u(n - j)
\]
\[
= u(n) - \binom{n - 1 - \alpha}{n - 1} u(0) - \sum_{j=1}^{n-1} \frac{1}{(j - \alpha)} \binom{j - \alpha}{j} u(n - j)
\]
\[
= u(n) - \binom{n - 1 - \alpha}{n - 1} u(0) - \sum_{j=1}^{n-1} \frac{\Gamma(j - \alpha + 1)}{j - \alpha \Gamma(j + 1) \Gamma(1 - \alpha)} u(n - j)
\]
\[
= u(n) - \binom{n - 1 - \alpha}{n - 1} u(0) - \sum_{j=1}^{n-1} \frac{j - \alpha - 1}{j} u(n - j)
\]
\[
= -\binom{n - 1 - \alpha}{n - 1} u(0) + \sum_{j=0}^{n-1} \binom{j - \alpha - 1}{j} u(n - j).
\]

Similarly, for $m=2$, (2.5) becomes
\[
\nabla^\alpha u(n) = \sum_{j=0}^{n-1} \binom{\alpha - 2}{j} (-1)^j \nabla^2 u(n - j) = \sum_{j=0}^{n-1} \binom{j - \alpha + 1}{j} \nabla^2 u(n - j)
\]
\[
= \sum_{j=0}^{n-1} \binom{j - \alpha}{j} \nabla u(n - j) - \binom{n - \alpha}{n - 1} \nabla u(n) \mid n = 0
\]
\[
= \sum_{j=0}^{n-1} \binom{j - \alpha - 1}{j} u(n - j) - \binom{n - 1 - \alpha}{n - 1} u(0).
\]

Now, we have for any $m$
\[
\nabla^\alpha u(n) = \sum_{j=0}^{n-1} \binom{j - \alpha - 1}{j} u(n - j) - \binom{n + m - 2 - \alpha}{n - 1} \nabla^m u(n) \mid n = 0.
\]

**Lemma 2.1** [3] Let $u(n)$ be a real valued function defined for $n \in \mathbb{N}_0^+$ and $f(n, r)$ be a function defined on $n \in \mathbb{N}_0^+, 0 \leq r < \infty$. Then for $n \geq 0$ and $0 < \alpha < 1$, if
\[
\nabla^\alpha u(n + 1) = f(n, u(n)),
\]
then
\[ u(n) = u(0) + \sum_{j=0}^{n-1} \left( \frac{n - j + \alpha - 2}{n - j - 1} \right) f(j, u(j)). \] (2.6)

**Lemma 2.2** [6] Let \( \alpha \in (0, 1) \) and \( u_n \) be a real valued function defined for \( n \in \mathbb{N}_0^+ \). Then
\[ \nabla^{-\alpha} [\nabla^{\alpha} u(n + 1)] = u(n + 1) - u(0). \]

**Lemma 2.3** (Discrete Gronwall’s Inequality) [12] Let \( y(n) \), \( a(n) \) and \( b(n) \) be any three nonnegative functions defined for \( n \in \mathbb{N}_0^+ \). If for \( n \in \mathbb{N}_0^+ \),
\[ y(n + 1) \leq y(0) + \sum_{j=0}^{n} [a(j)y(j) + b(j)], \]
then
\[ y(n) \leq y(0) \exp \left[ \sum_{j=0}^{n-1} a(j) \right] + \sum_{j=0}^{n-1} b(j) \exp \left[ \sum_{k=j+1}^{n-1} a(k) \right]. \]

3 MAIN RESULTS

Let \( u(n) \) be any function defined for \( n \in \mathbb{N}_0^+ \) and let \( f(n, r) \) be any function defined for \( n \in \mathbb{N}_0^+, 0 \leq r < \infty \). Then for \( 0 < \alpha < 1 \) and \( n \geq 0 \), consider a discrete fractional periodic boundary value problem as
\[ \nabla^{\alpha} u(n + 1) = f(n, u(n)), \quad u(0) = u(N + 1). \] (3.1)

**Definition 3.1** A function \( v(n) \) defined on \( \mathbb{N}_0^+ \) is said to be a lower solution of PBVP (3.1) if
\[ \nabla^{\alpha} v(n + 1) \leq f(n, v(n)), \quad v(0) \leq v(N + 1). \] (3.2)

Similarly a function \( w(n) \) defined on \( \mathbb{N}_0^+ \) is said to be an upper solution of PBVP (3.1) if
\[ \nabla^{\alpha} w(n + 1) \geq f(n, w(n)), \quad w(0) \geq w(N + 1). \] (3.3)

**Definition 3.2** Let \( r(n) \) be a solution of PBVP (3.1) on \( \mathbb{N}_0^+ \). Then \( r(n) \) is said to be a maximal solution of PBVP (3.1) if for every solution \( u(n) \) of PBVP (3.1), \( u(n) \leq r(n) \) for \( n \in \mathbb{N}_0^+ \). Similarly, \( p(n) \) is said to be minimal solution of PBVP (3.1) if for every solution \( u(n) \) of (3.1), \( p(n) \leq u(n) \) for \( n \in \mathbb{N}_0^+ \).

**Remark 1** Let \( v \) and \( w \) be lower and upper solutions of PBVP (3.1). Then for every solution of \( u(n) \) of PBVP (3.1), \( v(n) \leq u(n) \leq w(n) \), provided \( v(n_0) \leq u(n_0) \leq w(n_0) \).

**Theorem 3.1** Let \( f : \mathbb{N}_0^+ \times \mathbb{R} \rightarrow \mathbb{R}, v, w : \mathbb{N}_0^+ \rightarrow \mathbb{R} \) and
\[ \nabla^{\alpha} v(n + 1) \leq f(n, v(n)), \quad v(0) \leq v(N + 1) \]
\[ \nabla^{\alpha} w(n + 1) \geq f(n, w(n)), \quad w(0) \geq w(N + 1). \]

Further assume that \( f(n, r), r \in \mathbb{R} \) is strictly non decreasing in \( r \) for each \( n \). Then \( v(n) \leq w(n) \) provided \( v(0) \leq w(0) \).
Proof: If the statement is not true, then there exists a \( k \in \mathbb{N}_0^+ \) such that \( v(i) \leq w(i) \) for \( i = 0, 1, 2, ..., k \) and \( v(k + 1) > w(k + 1) \). Now

\[
f(k, w(k)) \leq \nabla^\alpha w(k + 1) = w(k + 1) - \left( \begin{array}{c} k - \alpha \\ k \end{array} \right) w(0) - \frac{\alpha}{j - \alpha} \left( \begin{array}{c} j - \alpha \\ j \end{array} \right) w(k + 1 - j)
\]

\[
\leq v(k + 1) - \left( \begin{array}{c} k - \alpha \\ k \end{array} \right) v(0) - \frac{\alpha}{j - \alpha} \left( \begin{array}{c} j - \alpha \\ j \end{array} \right) v(k + 1 - j)
\]

\[
= \nabla^\alpha v(k + 1) \leq f(k, v(k)).
\]

which is a contradiction since \( f(n, r) \) is strictly nondecreasing in \( r \) for fixed \( n \) and \( v(i) \leq w(i) \) for \( i = 0, 1, 2, ..., k \).

Hence \( v(n) \leq w(n) \) for all \( n \).

Also, \( v(0) \leq v(N + 1) \leq w(N + 1) \leq w(0) \).

Next, we prove a comparison theorem.

Theorem 3.2 Let \( m \in [\mathbb{N}_0^+, \mathbb{R}] \), \( f(n, u(n)) \) be non decreasing in \( u(n) \) and assume that

i. \( \nabla^\alpha m(n + 1) \leq f(n, m(n)) \) \hspace{1cm} (3.4)

ii. \( \nabla^\alpha u(n + 1) = f(n, p(n, u)) + \frac{p(n, u) - u}{1 + u^2}, \quad u(0) = u(N + 1) \) \hspace{1cm} (3.5)

has a solution \( u \) for every lower solution \( v \) of PBVP (3.1), where

\[ p(n, u) = \max(v(n), u). \]

Then \( m(0) \leq m(N + 1) \) implies that \( m(n) \leq u(n) \).

Proof: Suppose that \( u(n) \) is a solution of (3.5). For every lower solution \( v(n) \) of PBVP (3.1), if \( v(n) \leq u \), then \( p(n, u) - u = 0 \) and \( u \) is also a solution of PBVP (3.1).

When \( v(n) > u \), \( p(n, u) = v(n) \). Hence (3.5) takes the following forms

\[
\nabla^\alpha u(n + 1) = f(n, v(n)), \quad u(0) = u(N + 1), \quad \text{for } v(n) \leq u(n)
\]

\[
\nabla^\alpha u(n + 1) = f(n, v(n)) + \frac{v(n) - u}{1 + u^2}, \quad u(0) = u(N + 1), \quad \text{for } v(n) > u(n).
\]

In either case, we have

\[
\nabla^\alpha u(n + 1) \geq f(n, u(n)). \hspace{1cm} (3.6)
\]

Further let \( m(n) \leq u(n) \) for \( n = 0, 1, 2, ..., k \) and \( u(k + 1) < m(k + 1) \). From (3.4) and (3.6), we have \( m(n) \leq u(n) \).

\[
f(k, u(k)) \leq \nabla^\alpha u(k + 1) = u(k + 1) - \left( \begin{array}{c} k - \alpha \\ k \end{array} \right) u(0) - \frac{\alpha}{j - \alpha} \left( \begin{array}{c} j - \alpha \\ j \end{array} \right) u(k + 1 - j)
\]

\[
< m(k + 1) - \left( \begin{array}{c} k - \alpha \\ k \end{array} \right) m(0) - \frac{\alpha}{j - \alpha} \left( \begin{array}{c} j - \alpha \\ j \end{array} \right) m(k + 1 - j)
\]

\[
= \nabla^\alpha m(k + 1) \leq f(k, m(k)).
\]
which is a contradiction since \( f(n, r) \) is strictly nondecreasing in \( r \) for fixed \( n \) and \( m(i) \leq u(i) \) for \( i = 0, 1, 2, \ldots, k \).
Hence \( m(n) \leq u(n) \) for all \( n \).
Further, if \( r \) is the maximal solution of PBVP (3.1), then \( m(n) \leq u(n) \leq r(n) \).

**Theorem 3.3** Suppose that \( \nabla^\alpha m(n+1) \leq -M m(n) \) for \( M > 0 \) and \( m(0) \leq m(N+1) \), then \( m(n) \leq 0 \).

**Proof:** If it is false, then there exists a \( k \in \mathbb{N}_0^+ \) such that \( m(k) > 0 \) and \( m(n) \leq 0 \) for \( n < k \). Consider

\[
\nabla^\alpha m(k) = m(k) - \binom{k-1-\alpha}{k-1} m(0) + \sum_{j=1}^{k-1} \binom{j-\alpha-1}{j} m(k-j) \leq -M m(k)
\]

or \( (M+1)m(k) + \sum_{j=1}^{k-1} \binom{j-\alpha-1}{j} m(k-j) - \binom{k-1-\alpha}{k-1} m(0) \leq 0 \)

Since \( m(k) > 0 \), \( m(n) \leq 0 \) for \( n = 0, 1, 2, \ldots, k - 1 \), \( \binom{j-\alpha-1}{j} > 0 \) and \( \binom{j-\alpha-1}{j} < 0 \) implies that it is a contradiction. Hence \( m(n) \leq 0 \) for \( n \in \mathbb{N}_0^+ \).

Here we shall discuss the monotone iterative method to obtain extremal solutions for the periodic boundary value problem.

**Theorem 3.4** Let \( f(n, r) \) be any function defined for \( n \in \mathbb{N}_0^+ \) and \( 0 \leq r < \infty \). Let \( v(n) \) and \( w(n) \) be lower and upper solutions of PBVP (3.1) defined on \( \mathbb{N}_0^+ \) such that \( v(0) \leq w(0) \). Assume that equality holds in (3.2) and (3.3) for \( n = 0 \) and also suppose that

\[
f(n, r) - f(n, s) \geq -M(r-s), \quad n \geq 1 \tag{3.7}
\]

\[
f(0, r) - f(0, s) = -M(r-s), \quad n = 0 \tag{3.8}
\]

for \( v(0) \leq s \leq r \leq w(0) \) and \( M \geq 0 \). Then there exist monotone sequences \( \{v_m\}, \{w_m\} \) such that \( v_m \to v, w_m \to w \) as \( m \to \infty \), where \( v \) and \( w \) are minimal and maximal solutions of PBVP (3.1) respectively.

**Proof:** Let \( z_n : \mathbb{N}_0^+ \to \mathbb{R} \) be such that \( v_0(n) \leq z_n \leq w_0(n) \), consider the following fractional difference equation of order \( \alpha, 0 < \alpha < 1 \) associated with boundary conditions

\[
\nabla^\alpha u(n+1) = f(n, z_n) - M[u(n) - z_n], \quad u(0) = u(N+1). \tag{3.9}
\]

It is clear that, if \( z_n = u(n) \), \( u(n) \) is the unique solution of (3.9) on \( n \in \mathbb{N}_0^+ \). If \( z_n \neq u(n) \), the nonhomogeneous fractional difference equation (3.9) has unique solution \( u(n) \) [5]. In order to construct monotone sequences \( \{v_m\}, \{w_m\} \), we define a mapping \( A : \mathbb{R} \to \mathbb{R} \) such that \( Az_n = u(n) \). Now we prove that \( A \) satisfies the following properties.

(a). \( v_0(n) \leq Av_0(n), w_0(n) \geq Aw_0(n) \).

(b). \( A \) is monotone operator on \( [v_0, w_0] = \{u(n)/v(0) \leq u(n) \leq w(0) \} \).
To prove (a): Set \( A v_0(n) = v_1(n) \), where \( v_1(n) \) is the unique solution of (3.9) with \( z_n = v_0(n) \). Set \( p(n) = v_0(n) - v_1(n) \). Consider
\[
\nabla^\alpha p(n + 1) = \nabla^\alpha [v_0(n + 1) - v_1(n + 1)] \\
\leq f(n, v_0(n)) - f(n, v_0(n)) + M[v_1(n) - v_0(n)] \\
= -M[v_0(n) - v_1(n)] \\
or \nabla^\alpha p(n + 1) \leq -M p(n) \text{ and} \\
\nabla^\alpha p(1) = \nabla^\alpha [v_0(1) - v_1(1)] \\
= f(0, v_0(0)) + M[v_1(0) - v_0(0)] - f(0, v_0(0)) \\
= -M p(0).
\]

Also \( p(0) = v_0(0) - v_1(0) \leq v_0(N + 1) - v_1(N + 1) \) or \( p(0) \leq p(N + 1) \).

By using Theorem 3.3, we have \( p(n) \leq 0 \) or \( v_0(n) \leq v_1(n) = A v_0(n) \).

Similarly we can prove \( w_0(n) \geq A w_0(n) \).

To prove (b): Let \( k \in N_0^+ \) and \( z_k \) and \( z_{k+1} \in [v_0(n), w_0(n)] \) such that \( z_k \leq z_{k+1} \). Suppose that \( A \{ z_k(n) \} = v_k(n) \) and \( A \{ z_{k+1} \} = v_{k+1}(n) \). Take \( q(n) = v_k - v_{k+1} \).

Consider
\[
\nabla^\alpha q(n + 1) = \nabla^\alpha [v_k(n + 1) - v_{k+1}(n + 1)] \\
= f(n, z_k(n)) - M[v_k(n) - z_k(n)] - f(n, z_{k+1}(n)) + M[v_{k+1}(n) - z_{k+1}(n)] \\
= M[z_k(n) - z_{k+1}(n)] + [v_{k+1}(n) - v_k(n)] + f(n, z_k) - f(n, z_{k+1}) \leq -M q(n)
\]

and
\[
\nabla^\alpha q(1) = \nabla^\alpha [v_k(1) - v_{k+1}(1)] \\
= f(0, z_k(0)) - M[v_k(0) - z_k(0)] - f(0, z_{k+1}(0)) + M[v_{k+1}(0) - z_{k+1}(0)] \\
= M[z_k(0) - z_{k+1}(0)] + [v_{k+1}(0) - v_k(0)] + f(0, z_k) - f(0, z_{k+1}) \leq -M q(0).
\]

Also \( q(0) = v_k(0) - v_{k+1}(0) \leq v_k(N + 1) - v_{k+1}(N + 1) \) or \( q(0) \leq q(N + 1) \).

By using Theorem (3.3), we have \( q(n) \leq 0 \) or \( v_k(n) \leq v_{k+1}(n) = A v_k(n), \forall \ k \in N_0^+ \).

Similarly we can prove that \( w_k(n) \geq w_{k+1}(n), \forall \ k \in N_0^+ \).

i.e.,
\[
v_0(n) \leq v_1(n) \leq v_2(n) \leq \cdots \leq v_m(n) \leq w_m(n) \leq \cdots \leq w_2(n) \leq w_1(n) \leq w_0(n).
\]

As \( m \to \infty \), say \( v_m(n) \to v \) and \( w_m(n) \to w \) where \( v \) and \( w \) are any two functions defined on \( n \in N_0^+ \). Also \( v_m(n) \) and \( w_m(n) \) satisfy
\[
\nabla^\alpha v_m(n + 1) = f(n, v_{m-1}(n)) - M[v_m(n) - v_{m-1}(n)], \\
v_m(0) = v_m(N + 1), \quad (3.10) \\
\nabla^\alpha w_m(n + 1) = f(n, w_{m-1}(n)) - M[w_m(n) - w_{m-1}(n)], \\
w_m(0) = w_m(N + 1). \quad (3.11)
\]

As \( m \to \infty \), \( v_m(n) \to v \) and \( w_m(n) \to w \). Thus
\[
\nabla^\alpha v = f(n, v), \quad v(0) = v(N + 1), \quad (3.12) \\
\nabla^\alpha w = f(n, w), \quad w(0) = w(N + 1). \quad (3.13)
\]
Hence the functions \( v \) and \( w \) defined on \( n \in \mathbb{N}_0^+ \) are solutions of PBVP (3.1). Now to prove that \( v \) and \( w \) are minimal and maximal solutions of PBVP (3.1) respectively, it is sufficient to prove that

\[
v \leq u(n) \leq w
\]

(3.14)

if \( u(n) \) is any solution of (3.1) such that \( v(0) \leq u(n) \leq w(0) \). Let the statement

\[
v_k \leq u(n) \leq w_k
\]

(3.15)

be true. Let \( v_k \) satisfy (3.10) and consider \( r(n) = v_{k+1}(n) - u(n) \). Now

\[
\nabla^\alpha r(n+1) = \nabla^\alpha [v_{k+1}(n+1) - u(n+1)] \\
= f(n, v_k(n)) - f(n, u(n)) - M[v_{k+1}(n) - v_k(n)] \\
\leq -M[u(n) - u(n)] - M[v_{k+1}(n) - v_k(n)] = -Mr(n).
\]

and

\[
\nabla^\alpha r(1) = \nabla^\alpha [v_1(n+1) - u(1)] \\
= (f(n, v_k(0)) - f(n, u(0)) - M[v_{k+1}(0) - v_k(0)] \\
\leq -[M[u(0) - u(0)] - M[v_{k+1}(0) - v_k(0)] = -Mr(0).
\]

Also \( r(0) = v_{k+1}(0) - u(0) \leq v_{k+1}(N+1) - u(N+1) \) or \( r(0) \leq r(N+1) \).

By using Theorem (3.3), we have \( v_{k+1}(n) \leq u(n) \).

Similarly we can prove \( u(n) \leq w_{k+1}(n) \).

Therefore \( v_{k+1}(n) \leq u(n) \leq w_{k+1}(n) \). Hence the statement (3.15) is true for \( m = k + 1 \).

By the principle of mathematical induction, the statement (3.15) is true for every \( m \in \mathbb{N}_0^+ \). Thus \( v_m(n) \leq u(n) \leq w_m(n) \). Taking the limit as \( m \to \infty \), we get \( v \leq u(n) \leq w \) such that \( v(0) \leq u_n \leq w(0) \). Hence the functions \( v \) and \( w \) defined on \( n \in \mathbb{N}_0^+ \) are minimal and maximal solutions of PBVP (3.1) respectively.

**Remark 2** For \( n \in \mathbb{N}_0^+ \) and \( 0 < \alpha < 1 \),

\[
\binom{n + \alpha - 1}{n} + \binom{n + \alpha - 1}{n-1} = \binom{n + \alpha}{n}
\]

(3.16)

**Proof:**

\[
\binom{n + \alpha - 1}{n} + \binom{n + \alpha - 1}{n-1} = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)} + \frac{\Gamma(n + \alpha)}{\Gamma(\alpha + 1)\Gamma_n} \\
= \frac{\Gamma(n + \alpha)[1/n + 1/\alpha]}{\Gamma(\alpha)\Gamma_n} = \binom{n + \alpha}{n}.
\]

**Theorem 3.5** In addition to the hypothesis of Theorem 3.4, if we assume that

\[
f(n, r) - f(n, s) \leq M(r - s), \; n \geq 1 \quad ,
\]

(3.17)

\[
f(0, r) - f(0, s) = M(r - s), \; n = 0 \quad .
\]

(3.18)

for \( v_0(n) \leq s \leq r \leq w_0(n) \) and \( M \geq 0 \). Then \( v(n) = w(n) = u(n) \) is the unique solution of PBVP (3.1) such that \( v_0(n) \leq u(n) \leq w_0(n) \).
Proof: Since $v(n) \leq w(n)$, it is enough to prove that $v(n) \geq w(n)$. Take $s(n) = w(n) - v(n)$. Consider

\[
\nabla^\alpha s(n+1) = \nabla^\alpha [w(n+1) - v(n+1)] = f(n, w(n)) - f(n, v(n)) \leq M[w(n) - v(n)] = Ms(n).
\]

Clearly

\[
\nabla^\alpha s(1) = \nabla^\alpha [w(1) - v(1)] = f(0, w(0)) - f(0, v(0)) = M[w(0) - v(0)] = Ms(0).
\]

Since $\nabla^\alpha s(n+1) \leq Ms(n)$ and $\nabla^\alpha s(1) = Ms(0)$. Using Lemma 2.3, we get

\[
\nabla^{-\alpha} \{\nabla^\alpha s(n+1)\} \leq M \nabla^{-\alpha} s(n)
\]

\[
\Rightarrow s(n+1) - s(0) \leq M \left[ \sum_{j=1}^{n} \left( \frac{n - j + \alpha - 1}{n - j} \right) s(j) - \left( \frac{n - 1 + \alpha}{n - 1} \right) s(0) \right]
\]

\[
\Rightarrow s(n+1) - s(0) \leq M \left[ \sum_{j=0}^{n} \left( \frac{n - j + \alpha - 1}{n - j} \right) s(j) - \left( \frac{n + \alpha}{n} \right) s(0) \right]
\]

or

\[
\Rightarrow s(n+1) \leq \left[ 1 - M \left( \frac{n + \alpha}{n} \right) \right] s(0) + M \left[ \sum_{j=0}^{n} \left( \frac{n - j + \alpha - 1}{n - j} \right) s(j) \right]
\]

\[
\Rightarrow s(n+1) \leq s(0) + M \sum_{j=0}^{n} \left( \frac{n - j + \alpha - 1}{n - j} \right) s(j).
\]

Using Discrete Gronwall’s Inequality, we get

\[
s(n) \leq s(0) \exp \left[ M \sum_{j=0}^{n-1} \left( \frac{n - j + \alpha - 2}{n - j - 1} \right) \right].
\]

Since $s(0) = w(0) - v(0) \leq w(N+1) - v(N+1) \leq 0$, we get $s(n) \leq 0$. Thus $w(n) \leq v(n)$. Hence $v(n) = w(n) = u(n)$ is the unique solution of pbvp (3.1) such that $v_0(n) \leq u(n) \leq w_0(n)$.

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References


