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# Similarity Solutions of Unsteady Transonic Small Disturbance Equations

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The work is dedicated to Professor S. D. Nigam.

**Abstract:** Similarity solutions for a two-dimensional Riemann proplem for the unsteady transonic small disburance equations are obtained by the direct method.

**1. INTRODUCTION**

Self-similar solutions for the unsteady transonic small disburance equations assume significance as they describe an asymptotic behaviour of the larger family of solutions of initial/boundary value problems (Barenblatt (1979)). Recently Tesdall and Hunter (2002) treated this problem numerically to understand the asymptotic behaviour of Mach reflection of weak shock waves. The present work is devoted to exact analysis by the method (see Sachdev and Mayil Vaganan (1995)) which is a refinement of the direct method of Clarkson and Kruskal (1989).

## 2. DIRECT SIMILARITY METHOD

The equations governing the unsteady transonic small disterburance are

$$u_t + \left(\frac{1}{2}u^2\right)_x + u_y = 0, \qquad (1)$$

$$u_y - u - x = 0, \qquad (2)$$

Following the work of Clarkson and Kruskal (1989), we seek solutions of (1) and (2) in the form

$$u = a(x, y, t) + b(x, y, t)\, f(z, x), \qquad (3)$$

$$u = l(x,y,t) + s(x,y,t)\, g(z,x), \qquad (4)$$

$$z = z(x, t), \quad x = x(y, t),$$

where *z* and ξ are similarity variables.

We substitute (3)-(4) in (1 )-(2) and require the resulting equations in the following form of an ordinary differential equations governing the functions *f(z,* ξ) and *g(z,* ξ)

$$L_1 + L_2 f + L_3 g + L_4 g_z + f_x = 0, \qquad (5)$$

$$L_5 + L_6 f + L_7 f_z + L_8 f_x + L_9 f^2 + L_{10} g + L_{11} g x + f f_z = 0. \qquad (6)$$

The functions $L_n(z, x)$, $n = 1,2,. . . , 11$ are introduced according to

$$a_y - l_x = b x_y L_1(z, x), \qquad (7)$$

$$b_y = b x_y L_2(z, x), \qquad (8)$$

$$-s_x = b x_y L_3(z, x), \qquad (9)$$

$$-s z_x = b x_y L_4(z, x), \qquad (10)$$

$$a_t + a a_x + l_y = b^2 z_x L_5(z, x), \qquad (11)$$

$$b_t + a_x b + a b_x = b^2 z_x L_6(z, x), \qquad (12)$$

$$b z_t + a b z_x = b^2 z_x L_7(z, x), \qquad (13)$$

$$b z_t = b^2 z_x L_8(z, x), \qquad (14)$$

$$b b_x = b^2 z_x L_9(z, x), \qquad (15)$$

$$s_y = b^2 z_x L_{10}(z, x), \qquad (16)$$

$$s x_y = b^2 z_x L_{11}(z, x), \qquad (17)$$

We make use of the following simplifying assumptions to determine α, β and z

**Assumption 1:** If a (x, t) (or l (x, t)) is to be obtained from an equation of the form

$$a(x,t) = \tilde{a}(x,t) + b(x,t)L(z) \text{ (or } l(x,t) = \tilde{l}(x,t) + s(x,t)\,G(z),$$

then we may set L(z) = 0 (or G = 0).

**Assumption 2:** If b(x, t) (or s (x, t)) is to be obtained from an equation of the form $b(x,t) = \tilde{b}(x,t)L(z)$ (or $s(x,t) = \tilde{s}(x,t)\,G(z)$, then we may set L(z) = 1 (or G = 1).

**Assumption 3:** If *z(x, t)* is to be obtained from an equation of the form *f(z)* = $\tilde{z}$*(x, t),* where *f* is an invertible function, then we may set *f(z) = z.*

Integrating the equations (8) and (15) with respect to *y* and *x* and using the assumption 2 we obtain $L_2 = 0, L_9 = 0$ and $b_x = b_y = 0.$ Dividing (9) by (10) and (16) by (17), integrating the resulting equations with respect to *x* and *y,* and using assumption 2, we get $L_3 = L_{10} = 0$ and $s_x = s_y = 0$. Thus

$$b = b(t) \quad s = s(t). \tag{18}$$

Inserting (18) in (12), integrating with respect to *x* and applying assumption 1, we get $L_6 = 0$ and

$$a = L(y,t) - x\frac{b¢}{b}. \tag{19}$$

Substituting (18), (19) in (7), integrating with respect to *x,* and using assumption 1, we arrive at $L_1 = 0$ and

$$l = xL_y + C(y,t), \tag{20}$$

where *C(y, t)* is the function of integration.

Writing $\Lambda_{11} = \partial\Delta_{11}/\partial z$ in equation (17), integrating with respect to $x$, and applying assumption 3 we get $\Lambda_{11} = 1$ and

$$z = L(t) + \frac{\sigma}{\beta^2}\xi y x. \tag{21}$$

In view of (21), equations (10) and (14) require that $\Lambda_4 = -c^2$ and $\Lambda_8 = l$ and yield

$$\xi_t = l\frac{\sigma}{\beta}\xi y \tag{22}$$

$$\frac{\sigma}{\beta} = c\beta^{1/2} \tag{23}$$

Equations (22) and (23) together lead to .

$$\xi = y + lc\int \beta^{1/2} dt. \tag{24}$$

Inserting (19), (20) and (21), equations (13) and (11) become

$$L'(t) + x\frac{\sigma'}{\beta^2} - \frac{3\sigma\beta'}{\beta^3}x + \frac{L\sigma}{\beta^2} = \sigma\Lambda_7, \tag{25}$$

$$A_t - x\left[\frac{\beta''}{\beta} - \frac{\beta'^2}{\beta}\right] - \frac{\beta'}{\beta}\left[A - x\frac{\beta'}{\beta}\right] + xA_{yy} + C_y = \sigma\Lambda_5. \tag{26}$$

The fact that the left-hand side of equations (25), (26) and $z$ are linear in $x$ demand that $\Lambda_5$ and $\Lambda_7$ must be linear in $z$. We write $\Lambda_5 = qz$, $\Lambda_7 = rz$ into (25) and (26), and equate the coefficients of $x^0$, $x^1$, on both sides to obtain

$$A_t - \frac{\beta'}{\beta}A + C_y = q\sigma L(t), \tag{27}$$

$$2\frac{\beta'^2}{\beta^2} - \frac{\beta''}{\beta} + A_{yy} = q\frac{\sigma^2}{\beta^2}, \tag{28}$$

$$L'(t) + \frac{A\sigma}{\beta^2} = r\sigma L(t), \tag{29}$$

$$\sigma' - 3\sigma\frac{\beta'}{\beta} = r\sigma^2. \tag{30}$$

Equations (28) and (29) require that $A(y, t) = A(t)$. We consider the case where $q = r = 0$. Then it is easily verified that $\sigma = d$ and $\beta = b$, where $d$ and $b$ are arbitrary constants, satisfies (28) and (30); and equations (27) and (29) give

$$A' + h(t) = 0, \tag{31}$$

$$\frac{b^2}{c}L' + h(t) = 0, \tag{32}$$

from which we can express $A$ and $L$ in terms of $h$.

Now equations (19), (20), (21) and (24) yield

$$\alpha = -\int h(t)dt, \tag{33}$$

$$\lambda = p(t) + yh(t), \tag{34}$$

$$\zeta = -\frac{d}{b^2}\int h(t)dt + \frac{d}{b^2}x, \tag{35}$$

$$\xi = y + clb^{1/2}t. \tag{36}$$

Substituting (33)-(36) in (3)-(4)

$$u = -\int h(t) + bf(\zeta, \xi), \tag{37}$$

$$\upsilon = p(t) + yh(t) + dg(\zeta, \xi). \tag{38}$$

Inserting for $L_n(z,x)$, $n = 1,2,\ldots,11$, thus found, in equations (5)-(6), we obtain the following first order, nonlinear ordinary differential equations for $f$ and $g$:

$$-c^2 g'(z) + f'(x) = 0 \tag{39}$$

$$lf'(x) + g'(x) + ff'(z) = 0 \tag{40}$$

Substituting (37)-(38) in equations (1)-(2)

$$-\frac{dh}{b} f_z + b^{3/2} c l f_x + d f f_z - \frac{d}{b} f_z \int h\,dt + d g_x = 0 \tag{41}$$

$$-\frac{h}{b} f_z + l f_x + f f_z - \frac{1}{b} f_z \int h\,dt + g_x = 0 \tag{42}$$

By experience we can find that

$$h = ae^{-t} \tag{43}$$

Differentiating (39) and (40) with respect to $x$ and $z$ respectively,

$$\frac{1}{c^2} f_{xx} = g_{xz}, \tag{44}$$

$$l f_{xz} + f f_{zz} + f_z^2 + g_{xz} = 0. \tag{45}$$

On inserting (44), (45) becomes

$$l f_{xz} + f f_{zz} + f_z^2 + \frac{1}{c^2} f_{xx} = 0. \tag{46}$$

Taking $f(x,z) = F(z)$, where $z = z - kx$, equation (46) reduces to

$$\left[F + \left(\frac{k^2}{c^2} - kl\right)\right] F'' + F'^2 = 0. \tag{47}$$

We may seek a solution of (47) in the form

$$F(z) = kl - \frac{k^2}{c^2} + k_3 z^{1/2}, \tag{48}$$

and

$$F(z) = kl - \frac{k^2}{c^2} + k_3 z^{1/2}, \tag{49}$$

Substituting (49) in (39), we have

$$g(x,z) = -\frac{kk_3}{c^2}(3 - kx)^{1/2} + k_4 - \frac{1}{2}k_3^2 x, \tag{50}$$

Substituting (43), (49) and (50) in equations (37)-(38), we find that

$$u = ae^{-t} + bkl - \frac{k^2 b}{c^2} + bk_3\left[\frac{d}{b^2}(ae^{-t} + x) - k(y + clb^{1/2}t)\right]^{1/2}, \tag{51}$$

$$u = p(t) + ae^{-t}y - \frac{dkk_3}{c^2}\left[\frac{d}{b^2}(ae^{-t} + x) - k(y + lb^{1/2}t)\right]^{1/2}. \tag{52}$$

## 3. CONCLUSION

Application of similarity transformation methods to the system of two first order equations for $u(x, t)$ and $v(x, t)$, namely, (1)-(2) is difficult as there are three independent variables and this means that any similarity method has to be applied twice to (1)-(2) in order to reduce them to ordinary differential equations. In the present investigation, in the first application of the direct method, the system (1)-(2) is transformed again to a system of two partial differential equations (39)-(40). Then instead of seaking to reduce (39)-(40) to a system of ordinary differential equations, we solve (39)-(40) directly.. Thus equations (49)-(50) provide a solution to the system of two first order nonlinear partial differential equations (39)-(40) governing $f(z, x)$ and $g(z, x)$.

It is evident from (51) and (52) that $u(x, y, t)$ and $v(x, y, t)$ are travelling wave like solutions The work is in progress to reduce the system (39)-(40) to a system of ordinary differential equations to facilitate the determination of new solutions to (1)-(2). This work will be published elsewhere.

**REFERENCES**

[1] **Barenblatt, G. I.,** *Similarity, self-similarity and intermediate asymptotics,* Consultant Bureau, New York, 1979.

[2] **Clarkson, P. A.**and **Kruskal, M. D.,** *New Similarity Reductions of the Boussinesq Equations,* J. Math. Phys. **36,**2201-2213, 1989.

[3] **Sachdev, P. L.** and **B. Mayil Vaganan,** *Exact Free Surface Flows for Shallow Water Eqations: Part* ***II,*** *The Compressible Case,* Stud. Appl. Math. **94,**:57-76, 1995.

[4] **Tesdall, M.**and **Hunter, K.,** *Self-Similar Solutions For Weak Shock Reflection,* Siam J. Appl. Math. **63**, 42-61, 2002.