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FIGURATE SYSTEMS

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ABSTRACT :

In this paper we investigate the generalized properties of the systems of figurate sequences of all dimensions.System of all such systems itself forms a system. It is proved that minimum of two operations are sufficient to traverse the complete system; starting from any point of the system. *Associated A. P.s* (AAP) of the figurate numbers also form a system. Properties of such systems are studied at length. There exists a unique translator sequence for the complete system of figurate sequences of particular dimension. Finally, it is shown that the sequence of triangular numbers of every dimension plays the most important role in the generalized operation of forming sequences of partial sums of any given sequence.

1 INTRODUCTION :

Figurate number is a number which can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a regular polygon the number is called a polygonal number. A sequence of numbers formed in this manner is called a figurate sequence. Different figurate sequences are formed depending upon the dimension we consider. Each dimension gives rise to a system of figurate sequences which are infinite in number. We study the system of these systems and draw some important conclusions.

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Definition 1.0.1: *For a given sequence* $\{s_n\}$ *the sequence* $D_l = \{t_n\} = \{| s_{n+1} - s_n |\}$ *is called a first difference sequence of* $\{s_n\}$*.* $D_2 = \{| t_{n+1} - t_n |\}$ *is called the second difference sequence of* $\{s_n\}$*.*

In general, D_n will denote the n[th] difference sequence of the given sequence.

Let us start with a figurate sequence of triangular numbers of dim 2.

f_T: 1, 3, 6, 10, 15, 21,......
D_1 :*2,3,4,5,6, ...*

Then D_1 is an A.P. with common difference $d = 1$, where f_T denotes the figurate sequence of triangular numbers of dim 2 and D_1 denotes the first difference sequence of f_T.

Definition 1.0.2 : *Associated A.P.of a figurate sequence:* *An A.P. obtained by taking successive differences of the consecutive terms of a figurate sequence is called an associated A.P.(AAP) of the given figurate sequence.*

Ex 1: In the above example the first difference sequence 2, 3, 4, 5, 6... is the associated A.P.of the sequence of triangular numbers of dim 2.
Ex 2: Consider the sequence of square pyramidal numbers of dim 3, with the sequences of successive differences as given below.
f_s: 1, 5, 14, 30, 55,
D_1 : 4, 9, 16, 25,
D_2: *5, 7, 9, 11,*
Then, D_2 : 5, 7, 9, 11, is the associated A.P.

In this manner for every figurative sequence there exists an associated A.P.

Definition 1.0.3: *Level of a figurate sequence:* *Common differenre of the associated A.P. of a figurate sequence is called the level of the figurate sequenre.*

For example, level of all sequences of triangular numbers in all dimensions is 1. Similarly level of all pentagonal numbers is 3. In general level of all n-gonal figurate sequences is n - 2.

Systems of figurative sequences: Let us consider the system of figurative sequences of dim 2 .

system 1, *dim* = 2

f_T: 1, 3, 6, 10, 15, 21,.....	(Triangular numbers).
f_S : 1, 4, 9, 16, 25, 36,.......	(Square numbers).
f_P: 1, 5, 12, 22, 35, 51,......	(Pentagonal numbers).
f_H : 1, 6, 15, 28, 45, 66,......	(Hexagonal numbers).

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where, f_T denotes the sequence of triangular numbers of dim 2, f_S denotes sequence of square numbers of dim 2, f_P denotes sequence of pentagonal numbers of dim 2 and f_H denotes the sequence of hexagonal numbers of dim 2 etc.

Note: Triangular figurate sequence of dimension 1 is : 1, 1, 1, 1, 1... Ref.[3]. The corresponding system of associated A.P.s of the above system is,

D_T : 2, 3, 4, 5, 6,.....
D_S : 3, 5, 7, 9, 11,.....
D_P : 4, 7, I0, 13, I6,...
D_H : 5,9,13,17,21, ...

Properties of systems of figurative numbers: We give here some systems of figurative numbers for ready reference.

System 2, dim = 3
f_T: 1, 4, 10, 20, 35, 56,.....
f_S : 1, 5, 14, 30, 55, 91,.......
f_P: 1, 6, 18, 40, 75, 126,......
f_H : 1, 7, 22, 50, 95, 165,......

System 3, dim = 4
f_T: 1, 5, 15, 35, 70, 126,.....
f_S : 1, 6, 20, 50, 105, 196,.......
f_P: 1, 7, 25, 65, 140, 266,......
f_H : 1, 8, 30, 80, 175, 230,......
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system 4, *dim* = 5
f_T : 1, 6, 21, 56, 126, 242, ...
f_S : 1, 7, 27, 77, 182, 378,...
f_P : 1, 8, 33, 98, 238, 504, ...
f_H : 1, 9, 39, 119, 294, 524, ...

Properties of systems of figurate sequences: *Prop* 1 : An *n - gonal* figurate sequence of *dim. r* = sequence of partial sums of the corresponding (r - 1), dimensional *n - gonal figurate sequence.*
For example, Pentagonal figurate sequence of *dim.3* is f_P : 1, 6, 18, 40, 75,.... where as, Pentagonal figurate sequence of *dim.4* is 1, 1 + 6, 1 + 6 + 18, 1 + 6 + 18 + 40, i.e. 1, 7, 25, 65, etc.

***Result* 1.** Taking sequence of partial sums successively we can move from one *n - gonal* sequence in one dimension to the other *n - gonal* sequence in any other dimension. One can apply the inverse of this operation successively to move backwards along tbe same line.

***Prop* 2 :** In a system of figurate numbers columns of the system form A.P.s.

***Prop* 3 :** Sequence of common differences in the A.P.s formed in *Prop* 2 above, forms a sequence with first term 0 followed by the sequence of triangular numbers of that system.

***Prop* 4 :** The $(m - 1)^{th}$ difference sequence of the sequence formed in Prop 3 is again an A.P. whose first term is (m -1) and common difference is 1, where m is the dimension of the system.

Properties 2,3,4 can be easily verified from the given systems.

Lemma 1.0.4 : *Every system of figurate numbers has a unique A.P. associated with it whose first term* is *the dimension of the system and the common difference* is 1.

Proof. The proof of the lemma is constructive and follows from the above properties.

Definition 1.0.5 : *For a given system of figurate numbers columns of the* *system are in A.P. The sequence of the common differences of the A.P.s* is *called a* cd-sequence.

Definition 1.0.6 ***Associated A.P. of the system:*** *For every system of figurate numbers some successive difference sequence of the cd-sequence forms an A.P. This sequence* is *called an associated A.P. of the system.*

For the system of figurate numbers of dimension 3 the cd-sequence is; 0, 1, 4, 10, 20, and the second difference sequence of the cd-sequence *is;* D_2 = 2, 3, 4, 5,.... which is an A.P.of the system.

***Prop* 5 :** The associated A.P. of a system of figurate numbers of *dim r* is also the $(r - 2)^{th}$ difference sequence of the cd-sequence of *dim* (r - 1).

Definition 1.0.7 : *For a given n-gonal figurate sequence* $\{s_n\}$ *in dimension* r, *if there exists a sequence* $\{t_n\}$ *such that* $\{s_n + t_n\}$ *is an n + 1-gonal figurate sequence of dimension* r, *then* $\{t_n\}$ *is called a translator sequence of* $\{s_n\}$.

$\{-t_n\}$ is also the translator sequence.

Consider system 2 of figurate sequences of dimension 3. If we add a sequence 0, 1, 4, 10, 20, 35, 56,.... to any of the sequences of the above system, we get the next level sequence of same system.

This sequence has first term as 0 and remaining part is the triangular sequence of the system.

Proposition 1.0.8 : *Every system of figurate numbers has a unique translator sequence whose first term is* 0 *and remaining part is the triangular sequence of that system.*

Proof: The required translator sequence can always be constructed as above.

***Prop.* 5:** cd-sequence of the system of dimension r of figurate sequences is the translator sequence of that system.

The above proposition 1.0.7 leads to the following result.

***Result* 2:** In any figurate sequence of dimension r, every other figurate sequence can be generated by the figurate sequence of triangular numbers only.

It is clear from result 1 and result 2 that in the whole system of the systems of figurate sequences in all dimensions we can move

1) in the same level from one sequence to the other by taking partial sums of the triangular number sequence in dimension 2, successively.

2) in the same dimension from one figurate sequence to the other starting from the triangular number sequence in that dimension.

Thus, triangular number sequence of dimension 2 generates the complete system of figurate numbers in all dimensions.

Systems of associated A.P.s of figurate sequences :

Associated A.P.s of figurate sequences in a particular dimension form a system. For example, the associated A.P.system (AAP-system) for dimension 2 and 3 are as follows.

AAP-system 1, dim = 2

T_{ap} : 2, 3, 4, 5, 6, 7, ...

S_{ap} : 3, 5, 7, 9, 11, 13,......

P_{ap} : 4, 7, 10 ,13, 16, ...

H_{ap} : 5, 9, 13, 17, 21, ...

AAP-system 2, dim = 3

T_{ap} : 3, 4, 5, 6, 7, 8, ...

S_{ap} : 5, 7, 9, 11, 13,......

P_{ap} : 7, 10 ,13, 16, 19, ...

H_{ap} : 9, 13, 17, 21, 25, 29, ...

Similarly, we can have system of associated A.P.s for any dimension.

Properties of the system of associated A.P.s of figurate sequences :

***Prop* 1**: *AAP - system 1* $\supset$ *AAP - system 2* $\supset$ *AAP - system 3* $\supset$ ......
Note that we can obtain AAP-system 2 from AAP-system 1 just by

removing the first column of the AAP-system 1. In fact successive removal of the columns leads to the successive AAP-systems.

***Prop* 2 :** For given dimension "m" and level "*l*" precise associated A.P. can be obtained. First term of the AAP is m + (*l*- 1) (m - 1).

Once we get the first term, the complete figurative sequence can be obtained by adding "1" successively to the first term. Thus, the required A.P. is {(n- 1)1 + [m + (ℓ - 1) (m - 1)}.

***prop* 3:** The first difference sequence of the main diagonal of any AAP-system is an A.P. with common difference *d* = 2.

prop 4: In every AAP-system elements of each column except the first column can be obtained by adding the first element of the previous column to the first element of this column successively.

***prop* 5:** If the common difference of the AAP in a particular dimension is *d* (= ℓ) then the corresponding figurate sequence is the *(d* + 2)-gonal figurate sequence.

***Prop.* 6:** There exists a unique translator A.P.for every AAP- system which translates the given AAP in the system to the next AAP in that system.

For example, in the AAP system of dim = 2, the translator sequence is : 1, 2, 3, 4, 5,..... In the AAP-system of dim = 3, the translator sequence is : 2, 3, 4, 5, 6,.....

We can generalize the result as: *For an AAP-system of dimension* m *the translator sequence* is : m - 1, m, m + 1, m + 2, m + 3,....

***Equivalence relation on associated A.P.s* :**

We can define an equivalence relation on all the associated A.P.s as follows.

Let s and $s^{/}$ be any two AAPs. Let s **~** $s^{/}$ iff $s^{/}$ is obtained by translation from s.Clearly " **~** " is reflexive, symmetric and transitive.

Thus, above relation will divide all the AAPs in to disjoint equivalence classes.

The equivalence classes are; all triangular AAPs, all square AAPs, all pentagonal AAPs and so on.

Generalization of the process of partial summation :

Proposition 1.0.9 : *If* $\{s_n\} = s_1, s_2, s_3, s_4, ...s_n, ...$, is *any infinite sequence, then for* $r \geq 3$ *the sequence of* r^{th} *successive partial sums of* $\{s_n\}$ is,{ (*sum of first n triangular numbers of dimension* (*r* - *2*))s_1 + (*sum of first* (*n* - 1) *triangular numbers of dimension* (r - 2)) $\sum_{n=1}^{2} s_n$ + (*sum of first* (*n* - 2) *triangular numbers ofdimension(r* - 2)) $\sum_{n=1}^{3} s_n$ +...... + $\sum_{n=1}^{n} s_n$}

Proof: *We will prove the theorem by method of induction.*
Let $s_1, s_2, s_3, s_4, ...s_n, ... = s_n$ *be a given infinite sequence. Then, the sequence of first partial sums* is ;

$$s_1, s_1 + s_2, s_1 + s_2 + s_3, ... = \{ \sum_{i=1}^{n} s_i \}$$

The sequence of second partial sums is;

$$s_1, (s_1 + \sum_{i=1}^{2} s_i), (s_1 + \sum_{i=1}^{2} s_i + \sum_{i=1}^{3} s_i), ...$$

Again, taking the partial sums of the above sequence we get the sequence of third partial sums as; $\sum_{i=1}^{n} s_i$

s_1, $[s_1 + (s_1 + \sum_{n=1}^{2} s_n)]$, $[s_1 + (s_1 + \sum_{n=1}^{2} s_n) + (s_1 + \sum_{n=1}^{2} s_n + \sum_{n=1}^{3}$)],.....={(1 + 1 + 1 +.... n times) s_1 + (1 + 1 + 1 +.... (n - 1) times)$\sum_{n=1}^{2} s_n$ + (1 + 1 + 1 +.... (n - 2) times) + ... + $\sum_{n=1}^{n} s_n$}.

Let P(r) ≡ {(*sum of first n triangular numbers of dimension* (*r-2*))s_1 + (*sum of first* (*n-1*) *triangular numbers of dimension* (*r* - *2*)) $\sum_{n=1}^{2} s_n$ + (*sum of first* (*n* - *2*) *triangular numbers of dimension* (*r* - 2)) $\sum_{n=1}^{3} s_n$ +...... + $\sum_{n=1}^{n} s_n$}. *Then, P(3)* ≡ *sequence of* 3 *rd successive partial sum of the sequence* $s_1, s_2, s_3, s_4, ...$= { (*sum of first n triangular numbers of dimension*1)s_1 + (*sum of first* (*n* - *1*) *triangular numbers of dimension* 1) $\sum_{n=1}^{2} s_n$

+ (*sum of first* (*n* - 2) *triangular numbers of dimension* 1) $\sum_{n=1}^{3} s_n$ +...... + $\sum_{n=1}^{n} s_n$} = {(1 + 1 + 1 +.... n times) s_1 + (1 + 1 + 1 +.... (n - 1) times) $\sum_{n=1}^{2} s_n$ + (1 + 1 + 1 +.... (n - 2) times) + ... + $\sum_{n=1}^{n} s_n$}.
Thus,P(3) is true.

Let P(r) be true. Then, *sequence of* r^{th} *successive partial sum* is, {(*sum of first n triangular numbers of dimension* (*r-2*))s_1 + (*sum of first* (*n-1*) *triangular numbers of dimension* (*r* - *2*)) $\sum_{n=1}^{2} s_n$ + (*sum of first* (*n* - *2*) *triangular numbers of dimension* (*r* - 2)) $\sum_{n=1}^{3} s_n$ +...... + $\sum_{n=1}^{n} s_n$}.

If we take the next partial sum of the above sequence, obviously coefficients of s_I *get added successively and resulting coefficient of* s_I *will be the summation of n terms of the next dimensional triangular sequence. Similarly, we will get the coefficient of* $\sum_{n=1}^{2} s_n$ *and* $\sum_{n=1}^{3} s_n$ *i.e. by successive summation of the previous coefficients.*

Therefore, P(r + 1) ≡{(*sum of first n triangular numbers of dimension* (*r-1*))s_1 + (*sum of first* (*n-1*) *triangular numbers of dimension* (*r* - *1*)) $\sum_{n=1}^{2} s_n$ + (*sum of first* (*n* - *2*) *triangular numbers of dimension* (*r* - 1)) $\sum_{n=1}^{3} s_n$ +...... + $\sum_{n=1}^{n} s_n$}.

Hence, P(r) ⇒ *P(r* + 1).

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