A NOTE ON COMPLETE INTEGRAL OF NON-LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

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Abstract

The main purpose of this article is to review the work in [1], on finding a complete integral of a nonlinear partial differential equation of first order of the form $f(x, y, z, p, q) = 0$. Charpits equations are used for this purpose.

1 INTRODUCTION

A solution of a nonlinear partial differential equation (NPDE) of the form

$$f(x, y, z, p, q) = 0,$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ involving two arbitrary constants is called a complete integral [2], and it is of the form

$$F(x, y, z, a, b) = 0.$$

(1.2)

A general method of finding such complete integral (CI) involves a solution of Charpits equations [2, 3]

$$\frac{dx}{fp} = \frac{dy}{fq} = \frac{dz}{pfp + qfq} = -\frac{dp}{fx + pfz} = -\frac{dq}{fy + qfz}.$$  (1.3)

A solution of the form

$$g(x, y, p, q, a) = 0,$$

(1.4)

involving only one arbitrary constant $a$ can be combined with the given NPDE (1.1) to obtain $p$ and $q$ in the form

$$p = p(x, y, z, a), \quad q = q(x, y, z, a).$$
Using these and the relation $dz = p\,dx + q\,dy$ leads us to obtain a Complete Integral (CI) of the form $F(x, y, z, a, b) = 0$. This technique can be easily used in obtaining a $CI$ for a NPDE of the form, [1],

$$z = px + qy + f(p, q).$$

In section 2 we consider such an equation and prove a general theorem related to it. Two examples are given to illustrate the theorem.

2 A GENERAL THEOREM

Consider a NPDE of the form, [1],

$$z = px + qy + f(p, q) \quad (2.1)$$

Charpits equations give us, [1],

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{px + qy + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$ 

Clearly $p = \text{constant} = a \ (\text{say})$ is a solution part. This combined with (2.1) gives us

$$q = g(z - ax, y, a),$$

$$dz = adx + g(z - ax, y, a)dy. \quad (2.2)$$

Now consider the function,

$$z = ax + by + f(a, b). \quad (2.3)$$

Solving for $b$, we must get

$$b = g(z - ax, y, a) \quad (2.4)$$

Hence from (2.3) and (2.4) we have,

$$dz = adx + bdy = adx + g(z - ax, y, a)dy.$$

This shows that (2.3) is a solution of (2.1). Further $q = b$ is also an integral of (2.1). By eliminating $p$ and $q$ we obtain,

$$z = ax + by + f(a, b),$$

which is a $CI$ of the given NPDE, [1]. To get the complete integral of the given nonlinear partial differential equation, we should try to obtain a second solution.
say, \( h(x,y,z,p,q,a,b) \) of the Charpit’s equations. Then the elimination of \( p \) and \( q \) will give us the complete integral; [1]. However we may not be able to eliminate \( p \) and \( q \) from the two solutions. We illustrate this point by an example.

**Example 2.1:** Consider the partial differential equation,

\[
q^2 - qy - p = 0.
\]  

(Charpit’s equations give)

\[
\begin{align*}
\frac{dx}{-1} &= \frac{dy}{2q - y} &= \frac{dz}{-p + 2q^2 - qy} &= \frac{-dp}{0} = \frac{-dq}{-q}.
\end{align*}
\]

An obvious integral is,

\[
p = a
\]  

(2.6)

and further 

\[-dx = dq,
\]

giving 

\[
q = be^{-z}.
\]  

(2.7)

Elimination of \( p \) and \( q \) from equations (2.5), (2.6) and (2.7) obviously does not contain the variable \( z \) and so it cannot be a complete integral. Alternatively, using equation (2.5) we have from Charpit’s equations that,

\[
\begin{align*}
\frac{dz}{-p + 2q^2 - qy} &= \frac{-dq}{-q}, \\
\frac{dz}{q^2} &= \frac{dq}{q},
\end{align*}
\]

giving 

\[
2z + b - y(2z + b)^{\frac{1}{2}} - a = 0.
\]  

(2.8)

This is a function of \( z \) but it is still not a complete integral since with \( b = 0 \) we get on differentiating,

\[
2q - (2z)^{-\frac{1}{2}} = 0, \text{ and so, } q = \frac{2z}{2(2z)^{\frac{1}{2}} - y}.
\]

Further \( p = 0 \), and so, on substituting in equation (2.5) we get,

\[
q^2 - yq - p = \frac{4z^2 - 4yz(az)^{\frac{1}{2}} + 2y^2z}{[2(2z)^{\frac{1}{2}} - y]^2} \neq 0
\]

Hence equation (2.8) is not a solution of equation (2.5).

We now prove the following theorem, [1].
Theorem 2.1 Let

\[ f(x, y, z, p, q) = 0, \quad (2.9) \]

be a first order partial differential equation, [1]. Then there exist two solutions,

\[ g(x, y, z, p, q, a) = 0, \quad (2.10) \]
\[ h(x, y, z, p, q, b) = 0, \quad (2.11) \]

of Charpits equations, such that the eliminant

\[ F(x, y, z, a, b) = 0, \quad (2.12) \]

of \( p \) and \( q \) from equations (2.9), (2.10), (2.11) is a complete integral of equation (2.12).

PROOF: Let \( g(x, y, z, p, q, a) = 0 \) be an arbitrary solution of Charpits equations. Then from equations (2.9) and (2.10), we get, [1],
\[ p = p_1(x, y, z, a), \quad q = q_1(x, y, z, a), \]
and \( dz = p_1 dx + q_1 dy \) is integrable with integral
\[ F_1(x, y, z, a) = 0. \quad (2.13) \]
Solving equation (2.13) for \( a \) we get
\[ F_2(x, y, z, b) = a. \quad (2.14) \]

Then
\[ dF_2 = \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz = 0. \quad \text{Or} \quad dz = p_2(x, y, z, b) dx + q_2(x, y, z) dy, \]
with \( p_2 = \frac{\partial F_2}{\partial x} / \frac{\partial F_2}{\partial z} \) \( q_2 = -\frac{\partial F_2}{\partial y} / \frac{\partial F_2}{\partial z}. \)
The equation \( f(x, y, z, p_2, q_2) = 0 \) must be satisfied.

The equation (2.14) is a complete integral of equation (2.9). Thus for arbitrary function \( \phi \),
\[ \phi(p - p_2, q - q_2) = 0 = h(x, y, p, q, b) \quad (2.15) \]
is an equation of the required form (2.12). This follows from the fact that
\[ p = p_1(x, y, z, a) = p_2(x, y, z, b), \]
\[ q = q_1(x, y, z, a) = q_2(x, y, z, b), \]
satisfy this equation and also (2.9) and (2.10). This completes the proof of the theorem.

Example 2.2: Solve the equation, [1]

\[ 2py^2 - q^2z = 0. \]  

(2.16)

Solution: Charpits equations for (2.16) are

\[
\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z},
\]

i.e.

\[
\frac{dx}{2y^2} = \frac{dy}{-2qz} = \frac{dz}{2py^2 - 2q^2z} = \frac{-dp}{-pq^2} = \frac{-dq}{4py - q^2},
\]

from which we obtain

\[
\frac{dz}{2py^2 - 2q^2z} = \frac{dz}{-q^2z} = \frac{dp}{pq^2},
\]

\[2p = \text{constant} = \frac{1}{2}a^2 \text{ (say)}. \]  

(2.17)

Solving for \(q\) we get \(q = \frac{ay}{2}\), and so, \(zdz = \frac{1}{2}a^2dx + aydy\). Thus the complete integral is

\[ z^2 = a^2x + ay^2 + b. \]  

(2.18)

Solving this equation for \(a\) we get, \(a = -\frac{y^2 + (y^4 + 4z^2x - 4xb)^{\frac{1}{2}}}{2x}\).

Differentiating partially with respect to \(y\), we get

\[ q = \frac{y(y^4 + 4z^2x - 4xb)^{\frac{1}{2}} - y^3}{2xz}, [1] \]

Eliminating \(p\) and \(q\) from the equations (2.16), (2.17) and (2.19), we get \(z^2 = a^2x + ay^2 + b\), as the complete integral of (2.16).

3 AN APPLICATION

Although the above theorem tells us that given an equation (2.9) we can find two solutions (2.10) and (2.11) of Charpits equations such that the eliminant (2.12) of \(p\) and \(q\) from these three equations is a complete integral of equations (2.9). It does not help us to find a particular pair of such equations (2.10) and (2.11). Indeed in the method of proof we had to find a complete integral using equation (2.10) before we could find equation (2.11) and so the method of the proof is not of much use, if we are trying to find a complete integral, [1].

However, it may not be always possible to obtain the complete integral of (2.9)
from two solutions of (2.10) and (2.11) by eliminating p and q from the two solutions.
Hence the method of proof may not be useful for finding the complete integral. But it is always possible to obtain a complete integral by this method if

\[ f(x, y, z, p, q) = 0, \quad (3.1) \]

is symmetric, where by symmetric we mean that, [1]

\[ f(x, y, z, p, q) = f(y, x, z, q, p). \]

Suppose that

\[ g(x, y, z, p, q, a) = 0, \quad (3.2) \]

is a solution of Charpits equations which is not symmetric. Then by symmetry of equation (3.1) we have, [1]

\[ g(y, x, z, q, p, b) = 0 \quad (3.3) \]

is also a solution of Charpits equations. Now eliminate p and q from the equations (3.1), (3.2) and the equation \( dz = p \, dx + q \, dy \). The we obtain

\[ F(x, y, z, a, b) = 0 \quad (3.4) \]

which is a complete integral of equation (3.1).

Since \( a, b \) are arbitrary constants,

\[ F(y, x, z, b, a) = 0 \quad (3.5) \]

is also a complete integral of equation (3.1). Now solving equations (3.1) and (3.3) for \( p \) and \( q \) we get, [1]

\[ p = p_2(x, y, z, b), \quad q = q_2(x, y, z, b). \]

Then by the symmetry of equation (3.1) and the pair of equations (3.2) and (3.3) it follows that, \( dz = p_2 dx + q_2 dy \) has the integral \( F(y, x, z, b, a) = 0 \) where \( a \) is constant of integration. This is simply equation (3.5), which means that \( p \) and \( q \) satisfy the three equation (3.1), (3.2) and (3.3) simultaneously. Then we find the complete integral of the given equations (3.5 by elimination of p and q from the three equations by integrating only once, [1]. Of course the method will work only when at least one of (3.1) and (3.2) contains \( z \).

**Example 3.1:** Find the complete integral of the differential equation,

\[ p^3 x + q^3 y = z. \quad (3.6) \]
Solution: Charpits equations are
\[
\frac{dx}{3p^2x} = \frac{dy}{3q^2y} = \frac{dz}{3p^3x + 3q^3y} = -\frac{dp}{p^3 - p} = -\frac{dq}{q^3 - q}.
\]
We therefore have
\[
\frac{dx}{3x} = \frac{pdp}{1 - p^2}.
\]
On integrating and solving for \(p\) we get,
\[
p = (1 - a^2x^{-\frac{2}{3}})^{\frac{1}{2}}. \tag{3.7}
\]
By the symmetry we have
\[
q = (1 - b^2y^{-\frac{2}{3}})^{\frac{1}{2}}. \tag{3.8}
\]
Now eliminate \(p\) and \(q\) from equation (3.6), (3.7) and (3.8). We obtain
\[
\left(1 - a^2x^{-\frac{2}{3}}\right)^{\frac{3}{2}} x + \left(1 - b^2y^{-\frac{2}{3}}\right)^{\frac{3}{2}} y = z.
\]
Solve equations (3.6) and (3.7) to give,
\[
q = \left\{ \frac{z - \left(1 - a^2x^{-\frac{2}{3}}\right)^{\frac{3}{2}} x}{y} \right\}^{\frac{1}{3}}. \tag{3.8}
\]
The complete integral is then found by integrating
\[
dz = (1 - a^2x^{-\frac{2}{3}})^{\frac{3}{2}} dx + \left\{ \frac{z - \left(1 - a^2x^{-\frac{2}{3}}\right)^{\frac{3}{2}} x}{y} \right\}^{\frac{1}{3}} dy.
\]
As an example of the case when equations (3.1) and (3.2) do not contain the variable \(z\) explicitly we will consider the following example.

Example 3.2: Obtain the complete integral of the differential equation,
\[
p^3x + q^3y = 0. \tag{3.9}
\]
Solution: Charpits equations give
\[
\frac{dx}{3p^2x} = \frac{dy}{3q^2y} = \frac{dz}{3p^3x + 3q^3y} = -\frac{dp}{p^3} = -\frac{dq}{q^3}.
\]
We therefore have,
\[ \frac{dx}{3p^2x} = \frac{-dp}{p^3}, \quad \text{i.e.} \quad \frac{dx}{3x} = \frac{-dp}{p}, \]
and on integrating we have,
\[ \frac{1}{3} \log x = -\log p + \log a \quad \text{giving} \quad p = ax^{-\frac{1}{3}}. \quad (3.10) \]
By symmetry we also have,
\[ q = by^{-\frac{1}{3}}. \quad (3.11) \]
Eliminating \( p \) and \( q \) from equation (3.9), (3.10) and (3.11) we get, \( a^3x + b^3y = 0 \),
which is the complete integral integral of the given differential equation.

References

