MEAN VALUE THEOREM

S.R. Joshi
8, Karmayog, Tarak Colony, Opp. to Ram-krishna Ashram, Beed bye pass Road, Aurangabad - 431 517, M. S., India.
E-mail: pranav.coatings@gmail.com
and
M.R. Gosavi,
Dept. of Mathematics, Maharashtra Mahavidyalaya, Nilanga, 413521, Dist. Latur, M. S., India.
E-mail: mrgosavi11@yahoo.com

Abstract

The main purpose of this article is to elaborate and illustrate the proofs of the mean value theorems for complex valued functions given in [? ] in the form of all inequality, and then demonstrate their uses in some applications, including l’Hospital’s rule. This is achieved by using the concept of full covers and Cousin’s Lemma [? ].

1 INTRODUCTION

During the last five decades or so, several authors including L.C. Barret(1969), M.P. Diazin [? ], S. Reich [? ], V.K. Kulkarni [? ], etc... have been attracted towards the study of different aspects of the mean value theorem (MVT) of the elementary calculus. They pondered how to make its proof simple and elegant [? ], how to generalize it [? ], how to use it in proofs of other theorems [? , ? ], and how to prove its converse [? , ? ].

In this paper the authors have given the detail proofs of the three versions of MVT for complex valued functions in the form of an inequality occurred in [? ], and their usefulness in applications, particularly in proving l’Hospital’s rule. Examples are also given to support the theorems proved. The results proved mainly depend on the concept of a full cover of a closed interval and Cousin’s lemma [? , ? ].
2 PRELIMINARIES

In this section we give some useful definitions required in proving the results of Sections 3 and 4. We also state the classical Cauchy’s MVT and Cousin’s Lemma [?] used in proving the results of the next section.

Let \( \mathbb{R} \) and \( C \) stand for the sets of real and complex numbers respectively. Let \( I = [a, b] \) be a closed interval in \( \mathbb{R} \). If \( f \) is a real valued or complex valued function defined on \( I \), then we denote the length \((b - a)\) of \( I \) and the difference \( f(b) - f(a) \) by \(|I|\) and \( f(I) \) respectively.

**Definition 2.1** A family \( FC \) of closed subintervals of \( I \) is said to be a full cover of \( I \) if there exists a function \( \delta(x) > 0 \) for \( x \in I \), and for any subinterval \([u, v]\) of \( I \), we have

\[
v - u < \delta(x) \implies [u, v] \in FC \quad (2.1)
\]

where \( u < x < v \).

**Definition 2.2** A set \( A \) in \( \mathbb{R} \) is said to be null if given \( \epsilon > 0 \), there exist open intervals \( I_n \) such that

\[
A \subset \bigcup_{n=1}^{\infty} I_n, \text{ whenever } \sum_{j=1}^{\infty} |I_n| < \epsilon.
\]

For example any countable subset of \( \mathbb{R} \) is null. Also the Cantor set is null.

**Definition 2.3** Let \( A \subset \mathbb{R} \). We say that a property \( P \) holds nearly everywhere in \( A \) if it holds in \( A \setminus E \) (the set of all points in \( A \) which are not in \( E \)), where \( E \) is a countable subset of \( A \).

**Definition 2.4** A function \( f : I \to C \) is said to be absolutely continuous on \( I \), if for every \( \epsilon > 0 \), there exists \( \eta > 0 \), such that for any finite family \( \{I_k : k = 1, 2, 3, \ldots, n\} \) of non-overlapping subintervals of \( I \),

\[
\sum_{k=1}^{n} |I_k| < \eta \implies \sum_{k=1}^{n} |f(I_k)| < \epsilon. \quad (2.2)
\]

**Definition 2.5** Let \( A \subset \mathbb{R} \). We say that a property \( P \) holds almost everywhere (in short a.e.) in \( A \) if it holds in \( A \setminus E \), where \( E \) is a null subset of \( A \).

We now state two results without proof. One is the classical MVT for two real valued functions and the second is Cousin’s Lemma [?].

**Theorem 2.1** If \( f, g : I \to \mathbb{R} \) are continuous and differentiable on \((a, b)\), with \( g'(x) \neq 0 \) for all \( x \in (a, b) \), then there exists a point \( c \in (a, b) \) such that

\[
\frac{f(I)}{g(I)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \quad (2.3)
\]
Remarks: The assumption \( g'(x) \neq 0 \) for all \( x \in (a, b) \) implies that \( g' \) does not change sign in \( (a, b) \). That is \( g \) is either increasing or decreasing in \( (a, b) \). Note that there is no method to find the value of \( c \) satisfying (2.3).

Theorem 2.2 (Cousin’s Lemma)?. Let \( FC \) be a full cover for \( I = [a, b] \). Then each closed subinterval \( J \) of \( I \) has a partition whose subintervals lie in \( FC \).

3 THREE VERSIONS OF MEAN VALUE THEOREM

In this section we state and prove three versions of a mean value theorem for complex valued functions defined on \( I \), in the form of an inequality [?]. The author of [?] called these theorems as Mean, Meaner and Meanest value theorems respectively. Here the meaner the theorem, the stronger it is.

Consider the following assumptions related to two functions \( f : I \to C \), and \( g : I \to \mathbb{R} \).

(A1) \( g : I \to \mathbb{R} \) is increasing.

(A2) \( f \) and \( g \) satisfy the inequality \( |f'(x)| \leq g'(x) \) everywhere in \( (a, b) \) while

\[
\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b). \quad (3.1)
\]

(A3) \( f \) and \( g \) satisfy the inequality \( |f'(x)| \leq g'(x) \) nearly everywhere in \( I \), while \( f \) is continuous in \( I \).

(A4) \( f \) and \( g \) satisfy the inequality \( |f'(x)| \leq g'(x) \) a.e. in \( I \), while \( f \) is absolutely continuous on \( I \).

Theorem 3.1 Let the assumptions (A1) and (A2) hold. Then

\[ |f(J)| \leq g(J) \quad (3.2) \]

for every closed interval \( J \) of \( I = [a, b] \).

Proof: The proof of this theorem follows directly from the next theorem, by taking the countable set \( D \) as \( \{a, b\} \).

Theorem 3.2 Let the assumptions (A1) and (A3) hold. Then the inequality (3.2) holds.

Proof: Without loss of generality, we may take \( J = I = [a, b] \). Let \( |f'(t)| \leq g'(t) \) for all \( t \in A\setminus E \), where \( E = \{S_1, S_2, S_3, \ldots \} \) is a countable subset of \( J \). Let \( \epsilon > 0 \) be given and let \( F = \bigcup_{n=0}^{\infty} C_n \) be an infinite union of families of closed subintervals of \( J \), such that,
(i) $C_0$ is the family of all closed subintervals $K$ of $J$ such that $|f(K)| \leq g(K) + \epsilon |K|$, 

(ii) $C_n (n \geq 1)$ is a family of all closed subintervals $K$ of $J$ such that $s_n \in K$ and  

$$|f(K)| \leq \frac{\epsilon}{2^{n+1}}$$  

(3.3)  

We now show that $F$ is a full cover for $J$. If $x \in J \setminus D$, then by differentiability of $f$ and $g$ there exists $\delta(x) > 0$ such that for any closed subinterval $K = [u, v]$ of $J$ with $x \in [u, v]$ and $(v - u) < \delta(x)$, we have  

$$\left| \frac{f(v) - f(u)}{v - u} - f'(x) \right| \leq \frac{\epsilon}{2},$$  

(3.4)  

and  

$$\left| \frac{g(v) - g(u)}{v - u} - g'(x) \right| \leq \frac{\epsilon}{2}.$$  

(3.5)  

From these relations and the assumption (A3), we get  

$$|f(v) - f(u)| \leq |f'(x)||(v - u)| + \epsilon(v - u)/2$$  

$$\leq g'(x)(v - u) + \epsilon(v - u)/2$$  

$$\leq g(v) - g(u) + \epsilon(V - u).$$  

i.e.  

$$|f(K)| \leq g(K) + \epsilon(K).$$  

(3.6)  

Hence by (i) we conclude that $K \in C_0$. If $x \in D$, then for some $n, x = s_n$. Hence by continuity of $f$, there exists $\delta(s_n)$, such that for any $K = [u, v]$ in $J$, satisfying, $S_n \in [u, v]$, and $(v - u) < \delta(s_n)$, we have  

$$|f(v) - f(u)| \leq \frac{\epsilon}{2^{n+1}}.$$  

(3.7)  

This shows that $K \in C_n$. Thus $F$ is a full cover of $J$. Hence by Cousin’s Lemma, there exists a partition say $P = \{t_0, t_1, t_2, \ldots, t_p\}$ of $J$ such that $K_j \in F$, where $K_j = [t_{j-1}, t_j]$, for $j = 1, 2, 3, \ldots, p$. We observe that each $C_n$ with $n > 0$, contains at most two of the points of $K_j$. Hence  

$$|f(J)| = \sum_{j=1}^{p} f(K_j) \leq \sum_{j=1}^{p} |f(K_j)|$$  

(3.8)
The right hand side of (3.8) can be split up into two sums, depending on whether $K_j \in C_0$ or $K_j \in C_n (n > 0)$. Hence from (3.8) we get

$$|f(J)| \leq \sum_{K_j \in C_0} |f(K_j)| + \sum_{n=1}^{\infty} \sum_{K_j \in C_n} |f(K_j)|$$

Further using (3.6), (3.7) and increasing nature of $g$ we further get

$$|f(J)| \leq \sum_{j=1}^{p} (g(K_j) + \epsilon|K_j|) + 2 \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \leq g(J) + (|J| + 1)\epsilon.$$ 

But $\epsilon$ is arbitrary. Hence $|f(J)| \leq g(J)$, and the proof is complete.

The following examples will illustrate the theorems 3.1 and 3.2.

**Example 3.1** Let $f(x) = \sin x + i \cos x$ and $g(x) = x^3/3$ be two functions defined on $I = [1, n]$, where $n$ is any positive integer. Then it is easy to verify that the conditions (A1) and (A2) hold on $I$. Hence the inequality (3.2) holds for every closed subinterval $J$ of $I$. For example if $J = [\pi/2, \pi]$, then $|f(J)| = \sqrt{2}$ and $g(J) = 7\pi^3/24$. Clearly $|f(J)| \leq g(J)$. Note that if we take $I$ as $[0, \pi/2]$, then the condition (A1) does not hold on $I$, and hence the conclusion (3.2) does not hold on every subinterval $J$ of $I$. Similarly it can be verified that the functions $f(x) = 1 - \cos x$ and $g(x) = x^2/2$ defined on $I = [0, \pi/2]$ support Theorem 3.1.

**Example 3.2** Consider the following functions $f$ and $g$ defined on $[0,2]$.

$$f(x) = \begin{cases} 1/2 & 0 \leq x < 1, \\ x/2 & 1 \leq x < 3/2, \\ 3/2 - x/2 & 3/2 \leq x \leq 2, \end{cases}$$

and

$$g(x) = \begin{cases} x/2 & 0 \leq x < 1, \\ x & 1 \leq x < 3/2, \\ 3/2 & 3/2 \leq x \leq 2. \end{cases}$$

It can be observed that $g$ is an increasing function on $I$, $f'(x), g'(x)$ exist on $I \setminus E$, where $E = \{1, 3/2\}$ and $|f'(x)| \leq g'(x)$ for $x \in I \setminus E$. This shows that the conditions (A1) and (A3) of Theorem 3.2 are satisfied. It is easy to verify that the conclusion (3.2) of Theorem 3.2 holds for all subintervals $J$ of $[0,2]$.

**Theorem 3.3** Let the assumptions (A1) and (A4) hold. Then the relation (3.2) holds true for every subinterval $J$ of $I$.
Proof: As in the proof of Theorem 3.2 we take \( J = I = [a, b] \). Let \( |f'(t)| \leq g'(t) \) a.e. That is \( |f'(t)| \leq g'(t) \), for \( t \in J \setminus E \) where \( E \) is a null subset of \( J \). Let \( \epsilon > 0 \) be given. Since \( f \) is absolutely continuous on \( J \), there exists a positive number \( \eta \), such that

\[
\sum_{k=1}^{n} |I_k| < \eta \implies \sum_{k=1}^{n} |f(I_k)| < \epsilon
\]  

(3.9)

for every finite non-overlapping family \( \{I_k\} \) of subintervals of \( J \). Since \( E \) is a null subset of \( J \), there exists a sequence of open intervals \( I_n \), such that

\[
E \subset \bigcup_{n=1}^{\infty} |I_n| \text{ and } \sum_{n=1}^{\infty} |I_n| < \eta.
\]

As in Theorem 3.2 let \( F \) be an infinite union of families of closed subintervals of \( J \), satisfying the conditions (i) and (ii) mentioned in the proof of Theorem 3.2. Here also we show that \( F \) is a full cover for \( J \). For this purpose let \( x \in J \). There are two cases.

Case (i) : \( x \in J \setminus E \). In this case we have by following the argument as in the proof of Theorem 3.2, \( K \in C_0 \), whenever \( K = [u, v] \) is a subinterval of \( J \).

Case (ii) : Let \( x \in E \). Then there exists \( \delta(x) > 0 \) such that for any \( K = [u, v] \subset J \), satisfying \( x \in [u, v] \) and \((v - u) < \delta(x)\), we have \( K \subset I_n \) for some \( n > 0 \). Thus \( F \) is a full cover for \( J \). By Cousin’s Lemma there exists a partition \( P = \{t_0, t_1, t_2, \ldots, t_p\} \) of \( J \), for which each subinterval \( K_j = [t_{j-1}, t_j] \) belongs to \( F \). Observe that all the \( K_j \) contained in \( I_n \) have total length not exceeding \( |I_n| \). Hence

\[
\sum_{n=1}^{\infty} \sum_{K_j \subset I_n} |f(K_j)| < \epsilon.
\]  

(3.10)

This is so because the intervals \( K_j \) form a finite family of subintervals of \( J \) whose total length does not exceed \( \eta \). Now

\[
|f(J)| = \left| \sum_{j=1}^{p} f(K_j) \right| \leq \sum_{j=1}^{p} |f(K_j)| = \sum_{K_j \in C_0} |f(K_j)| + \sum_{n=1}^{\infty} \sum_{K_j \in C_n} |f(K_j)|.
\]

Here we have divided the sum \( \sum_{j=1}^{p} |f(K_j)| \) into two sums depending upon whether \( K_j \in C_0 \) or \( K_j \in C_n \). From this relation and using (3.6), (3.10) and positivity of \( g(K_j) \) we finally get

\[
|f(j)| \leq \sum_{K_j \in C_0} (g(K_j) + \epsilon |K_j|) + \sum_{n=1}^{\infty} \sum_{K_j \in C_n} |f(K_j)| \leq \sum_{j=1}^{p} g(K_j) + \epsilon \sum_{j=1}^{p} |K_j| + \epsilon \leq g(J) + (|J| + 1)\epsilon.
\]
Since $\epsilon$ is arbitrary, we have $|f(J)| \leq g(J)$, and the proof is complete. The following example supports the above theorem.

**Example 3.3** Let $I = [0, 1], A = \{a_n\}$, where $a_n = \frac{1}{2^n}, (n = 1, 2, 3, \ldots)$. Define two functions $f$ and $h$ on $I$ as follows:

$$f(x) = k, \text{ where } k \text{ is any fixed number, real or complex}$$

$$h(x) = \begin{cases} 0 & \text{if } x < \text{ every } a_n \\ \sum_{i=n}^{\infty} & \text{if } x \geq a_n \end{cases}$$

Clearly $h$ is a well defined and bounded function on $I$; and the range of $h$ is a countable set. It can also be observed that $h(0) = 0, h(1) = 1$ and

$$h(x) = \frac{1}{2^{n-1}} \text{ if } \frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}.$$

For example $h(1/16) = h(1/14) = h(1/15) = 1/8$.

Since $\lim_{x \to a_n^+} h(x) - \lim_{x \to a_n^-} h(x) = an \neq 0$, we conclude that $h$ is discontinuous at each $a_n$. Hence $h$ is continuous a.e. on $I$. Now define the function $g$ on $I$ as follows:

$$g(x) = \int_0^x h(t)dt, \ x \in I$$

Since $h$ is bounded on $I$ and continuous a.e. on $I, g$ is well defined on $I$. Further $g$ is increasing on $I$, because $h$ is a nonnegative on $I$. By fundamental theorem of calculus we have $g'(x) = h(x)$. Hence $g'(x)$ also exists a.e. on $I$. Since $f$ is clearly absolutely continuous on $I$, we see that the conditions ($A_1$) and ($A_4$) of Theorem 3.3 are satisfied. Hence the conclusion (3.2) holds. Direct verification of (3.2) is also easy since $|f'(x)| = 0$ and $g'(x) \geq 0$.

### 4 APPLICATIONS

Though the theorems proved in the previous section are in the form of an inequality, they can be used to prove the following known results.

**Theorem 4.1** If $f : I \to C$ satisfies $f'(x) = 0$ subject to one of the assumptions ($A2$), ($A3$) or ($A4$), then $f(x)$ is constant on $I$.

The proof of this theorem follows from Theorems 3.1,3.2 or 3.3, by taking $g(x) = 0$ on $I$. Note that in order to prove the constancy of $f$ on $I$, under the condition that $f'(x) = 0$ a.e. on $I$, we have to assume that $f$ is absolutely continuous function on $I$.

In the following theorem we are assuming that $f$ is also real valued.
Theorem 4.2 If \( f : I \to \mathbb{R} \) satisfies \( f'(x) \leq 0 (f' \geq 0) \) subject to one of the assumptions \((A2), (A3)\) or \((A4)\), then \( f \) is decreasing (increasing) in \( I \).

Proof: The function \( g(x) = 0 \) on \( I \) is increasing on \( I \) and if \([u, v] \subset I\), then by the conclusion of theorems 3.1 to 3.3, we have \( f([u, v]) = f(u) - f(v) \leq g([u, v]) = 0 \). Hence \( f \) is decreasing. The conclusion that \( f \) is increasing under the condition \( f' \geq 0 \) follows similarly by replacing \( f \) by \(-f\).

Lastly we prove a very well known and useful result known as l’Hospital’s Rule \([?]\) for finding the limit of \( \frac{f(x)}{g(x)} \) when \( x \to a \) under the condition that both \( f(x) \) and \( g(x) \) tend to either 0 or \( \infty \).

Theorem 4.3 Let \( f \) and \( g \) be two real valued functions defined on a domain \( X \) in \( \mathbb{R} \), containing a point \( a \). Let there exist an open interval \( I \) containing the point \( a \), such that the derivatives \( f' \) and \( g' \) exist and \( g'(x) \neq 0 \) in \( I, x \neq a \). Let

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = l. \tag{4.1}
\]

If either

\[
\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x), \tag{4.2}
\]

or \( \lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x). \tag{4.3} \)

Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = l. \tag{4.4}
\]

Proof: We prove the result for one sided limit from the left. Let \( \epsilon > 0 \) be given. By (4.1) there exists \( \delta > 0 \) such that \( J = (a - \delta, a) \subset I \) and

\[
\left| \frac{f'(x)}{g'(x)} - l \right| \leq \epsilon, a - \delta < x < a. \tag{4.5}
\]

Since \( g'(x) \neq 0 \) in \( J \) we may assume that \( g'(x) > 0 \) in \( J \). Hence \( g(x) \) is increasing in \( J \). Therefore by (4.5) we have

\[
(l - \epsilon)g'(x) \leq f'(x) \leq (l + \epsilon)g'(x), \text{ for } x \in J.
\]

Let \( F(x) = f(x) - (l + \epsilon)g(x) \) and \( G(x) = f(x) - (l - \epsilon)g(x) - f(x) \) for \( x \in J \). Then clearly \( F'(x) \leq 0 \) and \( G'(x) \leq 0 \) on \( J \). By Theorem 4.2, both \( F \) and \( G \) are decreasing on \( J \). Choose a point \( u < x \). Hence we have \( F(x) \leq F(u) \) and \( G(x) \leq G(u) \). That is

\[
f(x) - (l + \epsilon)g(x) \leq (l + \epsilon)g(u)
\]
and

\[(l - \epsilon)g(x) - f(x) \leq (l - \epsilon)g(u) - f(u).\]

Hence

\[(l - \epsilon)[g(x) - g(u)] \leq f(x) - f(u) \leq (l + \epsilon)[g(x) - g(u)].\]

Since \(g\) is strictly increasing, we have

\[
\left| \frac{f(x) - f(u)}{g(x) - g(u)} - l \right| \leq \epsilon
\]

(4.6)

As a first case let the relation (4.3) hold. Define

\[h(x) = \frac{1 - f(u)/f(x)}{1 - g(u)/g(x)}\]

for all \(x\) close to \(a\).

Then \(\lim_{x \to a^-} h(x) = 1\). Hence there exists \(\delta_1 \in (0, a - u)\) such that \(h(x) \geq 1/2\) and \(|l - h(x)| \geq \epsilon\), whenever \(a - \delta_1 < x < a\). Let \(u < a - \delta_1 < x < a\). Note that

\[
\frac{f(x)}{g(x)} h(x) = \frac{f(x) - f(u)}{g(x) - g(u)}, \quad 2h(x) \geq 1 \quad \text{and} \quad |1 - h(x)| < \epsilon,
\]

Hence by (4.6) we have,

\[
\left| \frac{f(x)}{g(x)} - l \right| = \left| \left( \frac{f(x)}{g(x)} - l \right) \times 1 \right|
\]

\[
\leq 2 \left| \frac{f(x)}{g(x)} h(x) - lh(x) \right|
\]

\[
= 2 \left| \frac{f(x) - f(u)}{g(x) - g(u)} - l + l \cdot h(x) \right|
\]

\[
\leq 2 \left| \frac{f(x) - f(u)}{g(x) - g(u)} - l \right| + 2|l||1 - h(x)|
\]

\[
< 2\epsilon + 2|l|\epsilon = 2(1 + |l|)\epsilon.
\]

Since \(\epsilon\) is arbitrary, we conclude that \(\lim_{x \to a^-} \frac{f(x)}{g(x)} = l\). Similarly we can show that \(\lim_{x \to a^+} \frac{f(x)}{g(x)} = l\). The case (4.2) can be dealt similarly.
References


